MID-SEMESTER EXAMINATION, 2ND SEMESTER, COMPLEX ANALYSIS, 2011-12

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1. One such holomorphic map is $f(z) = z$.

2. As f is holomorphic on a connected open set and assuming that it is not constant, we get f is an open map. But, the image of f is a subset of the set $\{(x, y) \in \mathbb{R}^2 \colon x = y^2\}$, so is clearly not an open set, which leads to a contradiction. So, f is constant.

3.a) The holomorphic map $\cos z$ is not 0 in a neighbourhood U of 0. So, $\tan z = \frac{\sin z}{\cos z}$ is holomorphic in U. So, the power series expansion of $\tan z$ is
$(z - \frac{z^3}{6} + \cdots)(1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots)^{-1}$
$= z(1 - \frac{z^2}{3} + \frac{z^4}{8} + \cdots)$.
So, the Laurent series expansion of $\frac{\tan z}{z^4}$ is
$\frac{1}{z^3} - \frac{1}{3}.\frac{1}{z} + \frac{1}{8}.z + \cdots$.
Hence, at 0, $\frac{\tan z}{z^4}$ has pole and the order of the pole is -3. The residue is $-\frac{1}{3}$.

b) The Laurent series of $\cos(\frac{1}{z})$ is $\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}.\frac{1}{z^{2n}}$. So, 0 is an essential singularity of $\cos(\frac{1}{z})$.
$\cos z = i$ implies $z = (2n\pi + \frac{\pi}{2}) + i\log(\sqrt{2} - 1)$ or $z = (2n\pi - \frac{\pi}{2}) + i\log(\sqrt{2} + 1)$.
Now, if we take $(2n\pi+\frac{\pi}{2})+i\log(\sqrt{2}-1)$, we get an infinite set of isolated points. So, there will be at least one n such that $|(2n\pi + \frac{\pi}{2}) + i\log(\sqrt{2} - 1)| > 1$, so for that n, we have $|\frac{1}{(2n\pi+\frac{\pi}{2})+i\log(\sqrt{2}-1)}| < 1$.
So, we get one $z \in B(0, 1)$, such that $\cos(\frac{1}{z}) = i$.

c) At all points except 0, $\frac{1-\cos z}{z^2}$ is holomorphic and its Laurent series expansion at 0 is
$\frac{1}{z^2}(1 - 1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \cdots) = \frac{1}{2} - \frac{z^2}{4!} + \cdots$.
So, at 0, f has removable singularity and one can define $f(0) = \frac{1}{2}$ to remove the singularity.

5. If f is holomorphic on U, then if we take the power series expansion, it would have a positive radius of convergence. Now, we take $a_0 = 0,\ a_n = n!\ \forall n \geq 1$. As we know that $n! \geq n^n e^{-n}$, so we have $(n!)^{\frac{1}{n}} \geq \frac{n}{e}$ and $\frac{n}{e}$ is unbounded as $n \to \infty$. So, radius of convergence of the power series $\sum a_n z^n$ is 0. Therefore, there is no holomorphic map on U, whose power series expansion is $\sum a_n z^n$.
For a holomorphic map $f = \sum a_n z^n$, we have $a_n = \frac{f^n(0)}{n!}$. Hence, if we take $c_0 = 0$ and $c_n = (n!)^2$, we shall not have any holomorphic map f with $f^n(0) = c_n$.

6. Let $f(z) = a_n(z - a)^n + a_{n+1}(z - a)^{n+1} + \cdots,\ n \geq 1$ be the power series expansion. So, $f'(z) = na_n(z - a)^{n-1} + (n + 1)a_{n+1}(z - a)^n + \cdots$.
Then, $g(z) = \frac{f'(z)}{f(z)}$
$= \frac{na_n(z-a)^{n-1}+(n+1)a_{n+1}(z-a)^n+\cdots}{a_n(z-a)^n+a_{n+1}(z-a)^{n+1}+\cdots}$.
$= \frac{na_n(z-a)^{n-1}(1+\cdots)}{a_n(z-a)^n(1+\cdots)}$

$= \frac{n}{z-a} + b_0 + b_1(z-a) + \cdots$.
Hence, residue of g at a is $n =$ order of zero of f at 0.

7. From Residue formula, we know that

$\int_\gamma \frac{f'}{f} = 2\pi i \sum m_i . Res_{z_i} \frac{f'}{f}$,
where γ is a closed curve in $B(a, 2r)$ homologous to 0 and $\frac{f'}{f}$ is a meromorphic function on $B(a, 2r)$ with only a finite no. of poles at points $z_1, \cdots, z_n$, none of which lie on γ and $m_i = W(\gamma, z_i)$ is the winding number.
In this case the cutve $\gamma(t) = a + re^{2\pi i t}$ is homologous to 0 in $B(a, 2r)$ and $\frac{f'}{f}$ has only a finite no. of poles, none of which lie on γ and which are in fact the zeroes of f. By Q6, $Res_{z_i} \frac{f'}{f}$ is the order of zero of f at z_i.
The no. m is $W(\gamma, z_i) = W(\gamma_i, z_i)$ where γ_i is a small circle around z_i contained in $B(a, 2r)$, because γ and γ_i are homotopic in $B(a, 2r)$. We can easliy compute that $W(\gamma_i, z_i) = 1$.
Therefore, $\int_\gamma \frac{f'}{f} = 2\pi i . \sum_i order_{z_i} f = 2\pi i .($ no. of zeroes of f in B(a,2r) counted according to multiplicity).